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Isotopic spin and coherent states

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Abstract. We review the concept of coherent states and extend the concept to a basis which transforms irreducibly under isotopic spin rotations. The construction is explicitly given for one-mode wavefunctions. The construction of a similar basis for fields can in principle be done by making use of the results in the present paper.

1. Introduction

The concept of coherent states was introduced by Schrödinger (1926) to study, in the coordinate representation, the minimum-uncertainty wavefunction of harmonic oscillators. Coherent state techniques are now widely employed in various branches of theoretical physics, and they can justify the study of classical equations of motion in quantum mechanics and quantum field theory (Hepp 1974, Klauder 1977, Perelomov 1977).

In the case of the electromagnetic field the coherent states, as discussed by Glauber (1963) and later developed in greater detail by others (Hepp 1974, Chung 1965, Rocca and Sirgue 1968, Kibble 1968, Roepstorft 1970), have been used, for example, in the study of optical coherence (Klauder and Sudarshan 1968). The infrared behaviour of scattering amplitudes in quantum electrodynamics has also been studied in terms of coherent states (Eriksson 1970 and references therein), and they are also useful when considering self-interacting boson fields (Hepp 1974, Eriksson and Skagerstam 1978).

Because of their useful properties many attempts have been made to generalise the concept of coherent states (for a general discussion see Bacry *et al* 1975). In Bhaumik *et al* (1976) it was noted that an extension to a situation in which an Abelian charge is involved is straightforward, and a complete basis of generalised coherent states in which the charge operator is diagonal can be constructed. The extension to field theory was discussed in Skagerstam (1978b), where the results were applied to the emission of soft charged pions from a 'classical' current.

At present accelerator energies the average number of particles produced is large, and therefore it might be useful to express the scattering operator in terms of generalised coherent states. In strong-interaction physics the charges involved are, however, non-Abelian in character (e.g. isospin). In Botke *et al* (1974) an attempt was made to construct coherent states for which the appropriate isospin operators can also be diagonalised. The pion field density operator was shown to take a simple form in the corresponding coherent states. A general framework for describing high-energy pion

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production has therefore been obtained. In Botke *et al* (1974) the isospin was, however, treated as a global variable, i.e. all pions were assigned the same momentum–space wavefunction. In the present paper we will give a similar construction for coherent states with definite isospin. We will restrict ourselves to one degree of freedom. The field theoretical case can be treated by using the methods in the present paper. We will return to this case elsewhere with a discussion of physical applications.

The states constructed in the present paper can, for example, be used in a semiclassical study of particles with isospin (see e.g. Arodz, 1978). These states can also be characterised uniquely by a minimal-dispersion relation, as is the case for conventional coherent states (Skagerstam 1978).

Our intention is to present the material in a self-contained manner. We cannot therefore avoid a repetition of the properties of conventional coherent states.

2. Fock space and isotopic spin

In what follows we give a short review of the one-mode Fock space construction with the corresponding representation of isotopic spin algebra. We introduce annihilation and creation operators

$$\boldsymbol{a} = (a_1, a_2, a_3), \qquad \boldsymbol{a}^+ = (a_1^+, a_2^+, a_3^+), \tag{1}$$

with commutation relations

$$[\boldsymbol{a}, \boldsymbol{a}] = 0, \qquad [\boldsymbol{a}^+, \boldsymbol{a}^+] = 0, \qquad [a_i, a_j^+] = \delta_{ij}. \tag{2}$$

We also introduce number operators

$$N_i = a_i^+ a_i, \tag{3}$$

with commutation relations

$$[N_i, N_j] = 0, \qquad [N_i, a_j] = -\delta_{ij} a_i, \qquad [N_i, a_j^+] = \delta_{ij} a_j^+. \tag{4}$$

The total number operator

$$N = N_1 + N_2 + N_3 = \boldsymbol{a}^+ \cdot \boldsymbol{a} \tag{5}$$

satisfies

$$[N, \boldsymbol{a}] = -\boldsymbol{a}, \qquad [N, \boldsymbol{a}^+] = \boldsymbol{a}^+. \tag{6}$$

Starting from the vacuum state $|0\rangle$, satisfying

$$\boldsymbol{a}|0\rangle = 0, \tag{7}$$

the eigenstates of the number operators (3) are

$$|n_1, n_2, n_3\rangle = (n_1! n_2! n_3!)^{-1/2} (a_1^+)^{n_1} (a_2^+)^{n_2} (a_3^+)^{n_3} |0\rangle. \tag{8}$$

Thus

$$N_i |n_1, n_2, n_3\rangle = n_i |n_1, n_2, n_3\rangle, \tag{9}$$

$$N |n_1, n_2, n_3\rangle = (n_1 + n_2 + n_3) |n_1, n_2, n_3\rangle. \tag{10}$$

Furthermore the states (8) are orthonormal,

$$\langle n_1', n_2', n_3' | n_1, n_2, n_3\rangle = \delta_{n_1' n_1} \delta_{n_2' n_2} \delta_{n_3' n_3}, \tag{11}$$

and complete,

$$\sum_{n_1,n_2,n_3} |n_1, n_2, n_3\rangle\langle n_1, n_2, n_3| = 1. \tag{12}$$

The effect of the annihilation and creation operators is as follows:

$$a_1|n_1, n_2, n_3\rangle = \sqrt{n_1}|n_1 - 1, n_2, n_3\rangle, \qquad a_1^+|n_1, n_2, n_3\rangle = \sqrt{n_1+1}|n_1+1, n_2, n_3\rangle. \tag{13}$$

Instead of the modes tied to the 1- and 2-directions we could equally well have used modes tied to the complex (±)-directions:

$$\begin{aligned} a_\pm &= (a_1 \pm \mathrm{i}a_2)/\sqrt{2}, & a_0 &= a_3, \\ a_\pm^+ &= (a_1^+ \mp \mathrm{i}a_2^+)/\sqrt{2}, & a_0^+ &= a_3^+. \end{aligned} \tag{14}$$

The isospin operators are

$$I_j = \mathrm{i}^{-1}\epsilon_{jkr}a_k^+a_r, \tag{15}$$

with commutation relations, obtainable from (2),

$$[I_j, I_k] = \mathrm{i}\epsilon_{jkr}I_r, \qquad \begin{cases} [I_j, a_k] = \mathrm{i}\epsilon_{jkr}a_r, \\ [I_j, a_k^+] = \mathrm{i}\epsilon_{jkr}a_r^+. \end{cases} \tag{16}$$

The squared isospin vector is

$$\boldsymbol{I}^2 = N^2 + N - A^+A, \tag{17}$$

where N is the number operator (5) and

$$A = \boldsymbol{a} \cdot \boldsymbol{a} = a_0a_0 + 2a_+a_-, \qquad A^+ = \boldsymbol{a}^+ \cdot \boldsymbol{a}^+ = a_0^+a_0^+ + 2a_+^+a_-^+. \tag{18}$$

A and A^+ satisfy the commutation relations

$$\begin{aligned} [\boldsymbol{I}, A] &= 0, & [N, A] &= -2A, \\ [\boldsymbol{I}, A^+] &= 0, & [N, A^+] &= 2A^+ \end{aligned} \tag{19}$$

and

$$\left.\begin{aligned} [a_i, A^+] &= 2a_i^+, \\ [A, a_i^+] &= 2a_i, \end{aligned}\right\} \qquad [A, A^+] = 4N + 6. \tag{20}$$

From the commutation relations (16) it can now be shown that the spherical harmonics $Y_m^l(\boldsymbol{a})^+$, defined as homogeneous lth degree polynomials in appendix 1, transform irreducibly under finite isospin rotations

$$R = \mathrm{e}^{-\mathrm{i}\boldsymbol{\theta}.\boldsymbol{I}} \tag{21}$$

according to

$$R^+Y_m^l(\boldsymbol{a})^+R^+ = D_{mm'}^l(R)Y_{m'}^l(\boldsymbol{a})^+ \tag{22}$$

(summation over repeated indices is to be understood), where $D_{m'm}^l(R)$ is the rotation matrix of the $(2l+1)$-dimensional irreducible representation of the rotation group.

Thus

$$Y_m^l(\boldsymbol{a})^+|0\rangle \tag{23}$$

is an eigenstate of $\boldsymbol{I}^2$ and I_3:

$$\boldsymbol{I}^2Y_m^l(\boldsymbol{a})^+|0\rangle = l(l+1)Y_m^l(\boldsymbol{a})^+|0\rangle, \qquad I_3Y_m^l(\boldsymbol{a})^+|0\rangle = mY_m^l(\boldsymbol{a})^+|0\rangle. \tag{24}$$

One way of proving the first relation (24) is to use (17) and the fact that for all l, m

$$[A, Y^l_m(\boldsymbol{a})^+]|0\rangle = 0. \tag{25}$$

3. Coherent states

The conventional coherent states are defined in terms of one-particle wavefunctions f by

$$|f\rangle = U(f)|0\rangle, \tag{26}$$

where $U(f)$ is the unitary operator

$$U(f) = \mathrm{e}^{\boldsymbol{a}^+ . f - f^* . \boldsymbol{a}} = \mathrm{e}^{-\frac{1}{2} f^* . f}\, \mathrm{e}^{\boldsymbol{a}^+ . f}\, \mathrm{e}^{-f^* . \boldsymbol{a}} \tag{27}$$

with the properties

$$U(f)^+ = U(f)^{-1} = U(-f), \qquad U(f)U(\boldsymbol{g}) = \mathrm{e}^{-\frac{1}{2}(f^* . \boldsymbol{g} - \boldsymbol{g}^* . f)} U(f + \boldsymbol{g}) \tag{28}$$

and

$$[\boldsymbol{a}, U(f)] = fU(f), \qquad [\boldsymbol{a}^+, U(f)] = -f^* U(f). \tag{29}$$

From (28) and (29) it follows that the coherent states are normalised eigenstates of the annihilation operator

$$\langle f|f\rangle = 1, \qquad \boldsymbol{a}|f\rangle = f|f\rangle. \tag{30}$$

The scalar product between two coherent states is easily obtained from the listed properties of the operator $U(f)$,

$$\langle \boldsymbol{g}|f\rangle = \mathrm{e}^{-\frac{1}{2}(f^* . \boldsymbol{g} - \boldsymbol{g}^* . f)}\, \mathrm{e}^{-\frac{1}{2}|f - \boldsymbol{g}|^2}, \tag{31}$$

where the first factor is a phase factor. Thus coherent states are never orthogonal.

The number operator of the mode described by the wavefunction f is

$$N_f = (\boldsymbol{a}^+ . f)(f^* . \boldsymbol{a})/f^* . f. \tag{32}$$

In the coherent state $|f\rangle$ corresponding to this mode it has the expectation value

$$\langle f|N_f|f\rangle = f^* . f. \tag{33}$$

The eigenstates of N_f,

$$N_f|n_f\rangle = n_f|n_f\rangle, \tag{34}$$

are

$$|n_f\rangle = \frac{(a_f^+)^{n_f}}{\sqrt{n_f!}}|0\rangle; \qquad a_f^+ = \frac{\boldsymbol{a}^+ . f}{\sqrt{f^* . f}}. \tag{35}$$

In terms of these $|f\rangle$ has the expansion

$$|f\rangle = \mathrm{e}^{-\frac{1}{2} f^* . f} \sum_{n_f = 0} \frac{(f^* . f)^{\frac{1}{2} n_f}}{\sqrt{n_f!}} |n_f\rangle. \tag{36}$$

Thus

$$|\langle n_f|f\rangle|^2 = \mathrm{e}^{-f^* . f}(f^* . f)^{n_f}/n_f!, \tag{37}$$

i.e. we have a Poisson distribution.

The expansion of $|f\rangle$ in terms of the Fock basis (8) is

$$|f\rangle = e^{-\frac{1}{2} f^* \cdot f} \sum_{\substack{n_1, n_2, \\ n_3 = 0}} \frac{(f_1)^{n_1}(f_2)^{n_2}(f_3)^{n_3}}{(n_1! n_2! n_3!)^{1/2}} |n_1 n_2 n_3\rangle, \tag{38}$$

implying that

$$|\langle n_1, n_2, n_3 | f\rangle|^2 = \prod_{j=1}^{3} \frac{e^{-|f_j|^2} |f_j|^{2n_j}}{n_j!}, \tag{39}$$

i.e. we have independent Poisson distributions of the three modes.

The coherent states form an over-complete basis. From (38) one easily shows the completeness relation

$$1 = \int d^6 f |f\rangle\langle f|, \tag{40}$$

where the measure is

$$d^6 f = \prod_{j=1}^{3} d^2 f_j, \qquad d^2 f_j = \frac{1}{\pi} d\,\mathrm{Re}\{f_j\}\, d\,\mathrm{Im}\{f_j\}. \tag{41}$$

In terms of the basis

$$|n_0, n_+, n_-\rangle = (n_0! n_+! n_-!)^{-1/2} (a_0^+)^{n_0} (a_+^+)^{n_+} (a_-^+)^{n_-} |0\rangle \tag{42}$$

derived from the representation (14), the expansion of a coherent state $|f\rangle$ is (in analogy with (38))

$$|f\rangle = e^{-\frac{1}{2} f^* \cdot f} \sum_{n_0, n_\pm = 0} \frac{(f_0)^{n_0}(f_+)^{n_+}(f_-)^{n_-}}{(n_0! n_+! n_-!)^{1/2}} |n_0, n_+, n_-\rangle, \tag{43}$$

where

$$f_\pm = (f_1 \mp i f_2)/\sqrt{2}. \tag{44}$$

The corresponding particle distribution is

$$|\langle n_0, n_+, n_- | f\rangle|^2 = \prod_{j=0,\pm} \frac{e^{-|f_j|^2} |f_j|^{2n_j}}{n_j!}. \tag{45}$$

The inverted relation corresponding to (43) is

$$|n_0, n_+, n_-\rangle = (n_0! n_+! n_-!)^{-1/2} \int d^6 f\, e^{-\frac{1}{2} f^* \cdot f} (f_0^*)^{n_0} (f_+^*)^{n_+} (f_-^*)^{n_-} |f\rangle. \tag{46}$$

4. Isospin and number operator basis

We can now use the fact that A^+ creates a pair in an isosinglet (see e.g. equation (19)) and introduce states which are eigenstates of $\boldsymbol{I}^2$ and I_3 as well as of the total number operator (5):

$$|l, m; l+2n\rangle = N_{l,n} (A^+)^n Y^l_m(\boldsymbol{a})^+ |0\rangle, \tag{47}$$

$$\begin{aligned} \boldsymbol{I}^2 |l, m; l+2n\rangle &= l(l+1) |l, m; l+2n\rangle, \\ I_3 |l, m; l+2n\rangle &= m |l, m; l+2n\rangle, \\ N |l, m; l+2n\rangle &= (l+2n) |l, m; l+2n\rangle. \end{aligned} \tag{48}$$

Here $N_{l,n}$ is a normalisation constant,

$$N_{l,n}^{-2} = \langle 0| Y_0^l(\boldsymbol{a}) A^n (A^+)^n Y_0^l(\boldsymbol{a})^+ |0\rangle, \tag{49}$$

which is easily evaluated by means of equations (20), (A1.6) and (A1.7). Inserting the result into (47) we have

$$|l, m; l+2n\rangle = \left(\frac{2^l(n+l)!}{(2n+2l+1)!n!}\right)^{1/2} (A^+)^n Y_m^l(\boldsymbol{a})^+|0\rangle. \tag{50}$$

A similar construction was recently discussed in Bartnik and Rzażewski (1977). If (50) is expressed in the basis (42), which are already eigenstates of N and I_3, then we have the constraints

$$l+2n = n_0 + n_+ + n_-, \qquad m = n_+ - n_-. \tag{51}$$

The expansion coefficients can be read from equations (47), (48) and (A1.2) as

$$|l, m; l+2n\rangle = \sum_{k=0}^{n+(l-|m|)/2} c_k^{(l,m,n)} \left| l+2n-|m|-2k; \frac{m+|m|}{2}+k; \frac{m-|m|}{2}+k \right\rangle, \tag{52}$$

where

$$c_k^{(l,m,n)} = (-1)^m 2^{k+\frac{1}{2}(l-m)} \times \left(\frac{(n+l)!(2l+1)!(l+m)!(l-m)!(l+2n-m-2k)!(m+k)!k!n!}{(2n+2l+1)!}\right)^{1/2}$$
$$\times \sum_{j=0}^{k} \frac{(-1)^j 2^{-2j}}{(k-j)!(n+j-k)!(l-m-2j)!j!(m+j)!}, \qquad m \geq 0 \tag{53}$$

and

$$c_k^{(l,m,n)} = (-1)^m c_k^{(l,-m,n)}, \qquad m \leq 0. \tag{54}$$

The inverted relation (52) is

$$|n_0, n_+, n_-\rangle = \sum_{l=0}^{n_0+n_++n_-} c_{(n_++n_--|n_+-n_-|)/2}^{(l,n_+-n_-,n_0+n_++n_-)} |l, n_+-n_-; n_0+n_++n_-\rangle. \tag{55}$$

5. Isospin decomposition of coherent states

We now wish to relate $|f\rangle$ and $|l, m; l+2n\rangle$ to each other, i.e. to make an isospin decomposition of $|f\rangle$. Starting from the definition of $|f\rangle$, (26) and (27),

$$|f\rangle = \mathrm{e}^{-\frac{1}{2} f^* \cdot f}\, \mathrm{e}^{a^+ \cdot f}|0\rangle, \tag{56}$$

we can now use (A1.9) to achieve such a decomposition:

$$|f\rangle = \mathrm{e}^{-\frac{1}{2} f^* \cdot f} \sum_{l=0}^{\infty} \phi_l((\boldsymbol{f}\cdot\boldsymbol{f})A^+) \sum_{m=-l}^{l} Y_m^l(\boldsymbol{f}) Y_m^l(\boldsymbol{a})^+|0\rangle. \tag{57}$$

Using (50) and (A1.10) we obtain the result

$$|f\rangle = \mathrm{e}^{-\frac{1}{2} f \cdot f} \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(\frac{2^l(n+l)!}{(2n+2l+1)!n!}\right)^{1/2} (\boldsymbol{f}\cdot\boldsymbol{f})^n Y_m^l(\boldsymbol{f}) |l, m; l+2n\rangle. \tag{58}$$

The inverse of this relation is

$$|l, m; l+2n\rangle = \left(\frac{2^l(n+l)!}{(2n+2l+1)!n!}\right)^{1/2} \int d^6f\, e^{-\frac{1}{2}f^*\cdot f}(f^*\cdot f^*)^n Y^l_m(f)^*|f\rangle. \tag{59}$$

The probabilities connected to the expansion (58) are

$$|\langle l, m; l+2n|f\rangle|^2 = e^{-f^*\cdot f}\frac{2^l(n+l)!}{(2n+2l+1)!n!}|f\cdot f|^{2n} Y^l_m(f) Y^l_m(f)^*. \tag{60}$$

The following partial sums are of interest:

$$\sum_{m=-l}^{l} |\langle l, m; l+2n|f\rangle|^2 = e^{-f^*\cdot f}\frac{2^l(n+l)!}{(2n+2l+1)!n!}|f\cdot f|^{2n+l}(2l+1)P_l\left(\frac{f^*\cdot f}{|f\cdot f|}\right), \tag{61}$$

$$\sum_{n=0}^{\infty} |\langle l, m; l+2n|f\rangle|^2 = e^{-f^*\cdot f}\phi_l(|f\cdot f|^2)|f\cdot f|^l\left|Y^l_m\left(\frac{f}{\sqrt{f\cdot f}}\right)\right|^2, \tag{62}$$

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l} |\langle l, m; l+2n|f\rangle|^2 = e^{-f^*\cdot f}\frac{(f^*\cdot f)^n}{n!}, \tag{63}$$

$$\sum_{n=0}^{\infty}\sum_{m=-l}^{l} |\langle l, m; l+2n|f\rangle|^2 = e^{-f^*\cdot f}\phi_l(|f\cdot f|^2)|f\cdot f|^l(2l+1)P_l\left(\frac{f^*\cdot f}{|f\cdot f|}\right). \tag{64}$$

To obtain (63) we used

$$\frac{2^l(n+l)!}{(2n+2l+1)!n!}|f\cdot f|^{2n} = \frac{1}{2\pi i}\oint_C dz\frac{\phi_l(z|f\cdot f|^2)}{z^{n+1}} \tag{65}$$

in (61), where the contour C encloses $z=0$. We also summed over l and applied (A1.11).

We can sum (63) over n or we can sum (64) over l using (A1.11) to obtain

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=0}^{\infty} |\langle l, m; l+2n|f\rangle|^2 = 1. \tag{66}$$

From (58) we can now extract eigenstates of $\boldsymbol{I}^2$, I_3 and A which, however, are not eigenstates of the total number operator

$$|\xi; l, m\rangle = N_l(|\xi|)\sum_{n=0}^{\infty}\left(\frac{2^l(n+l)!}{(2n+2l+1)!n!}\right)^{1/2}\xi^n|l, m; l+2n\rangle, \tag{67}$$

such that

$$\boldsymbol{I}^2|\xi; l, m\rangle = l(l+1)|\xi; l, m\rangle, \tag{68}$$

$$I_3|\xi; l, m\rangle = m|\xi; l, m\rangle, \tag{69}$$

$$A|\xi; l, m\rangle = \xi|\xi; l, m\rangle. \tag{70}$$

N_l is a normalisation factor given by the expression

$$N_l(|\xi|) = (\phi_l(|\xi|^2))^{-1/2}. \tag{71}$$

A completeness relation for the generalised coherent states (67) can be derived (see appendix 2),

$$\sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int \frac{d^2\xi}{2\pi} \Phi_l(|\xi|)|\xi; l, m\rangle\langle\xi; l, m| = 1, \tag{72}$$

where

$$\Phi_l(x) = \phi_l(x^2)k_l(x)x^{l+1} \tag{73}$$

and $k_l(\xi)$ is a spherical modified Bessel function. The relation (72) has a very close analogy to the completeness relation for coherent states with an Abelian charge (Bhaumik *et al* 1976, Skagerstam 1978b). As discussed in appendix 3, a projection technique similar to the one discussed in Bhaumik *et al* (1976) and Skagerstam (1978b) can be constructed by means of which the state (67) can be obtained from coherent states $|f\rangle$ by a suitable integration over the group manifold. We obtain

$$\int_0^{2\pi} \frac{d\alpha}{2\pi} \int_0^{\pi} \frac{d\beta \sin\beta}{2} Y^l_m(\beta, \alpha)^* |R(\alpha, \beta, 0)f\rangle = C(f)|\boldsymbol{f}\cdot\boldsymbol{f}; l, m\rangle, \tag{74}$$

where $C(f)$ is given by

$$C(f) = e^{-\frac{1}{2}f^*\cdot f}(N_l(|\boldsymbol{f}\cdot\boldsymbol{f}|))^{-1}(Y^l_0(f)/(2l+1)^{1/2}). \tag{75}$$

The states (67) can furthermore easily be related to the coherent states defined by equation (56):

$$|f\rangle = e^{-\frac{1}{2}f^*\cdot f} \sum_{l=0}^{\infty} (\phi_l(f^*\cdot f^* f\cdot f))^{-1/2} \sum_{m=-l}^{l} Y^l_m(f)|\boldsymbol{f}\cdot\boldsymbol{f}; l, m\rangle. \tag{76}$$

Finally we compute the overlap of two generalised coherent states (67) using (71) and (A1.10):

$$\langle\xi; l, m|\xi'; l', m'\rangle = \delta_{ll'}\delta_{mm'}\phi_l(\xi^*\xi')/(\phi_l(|\xi|^2)\phi_l(|\xi'|^2))^{1/2}. \tag{77}$$

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Appendix 1. Spherical harmonics

For our purposes it will be convenient to introduce spherical harmonics as homogeneous polynomials of the components of a unit vector (the expression A1.2 can be derived by making use of a Laplace integral for associated Legendre polynomials (MacRobert (1948)):

$$e = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{A1.1}$$

$$Y^l_m(e) = (-1)^m[(2l+1)(l-m)!(l+m)!]^{1/2}2^{-m} \times \sum_{k=0}^{(l-m)/2} \frac{2^{m/2-k}}{(l-m-2k)!k!(m+k)!} e_0^{l-m-2k} e_+^{m+k} e_-^k, \qquad m \geqslant 0 \tag{A1.2}$$

and

$$Y^l_{-m}(e) = (-1)^m Y^l_m(e)^*, \tag{A1.3}$$

where

$$e_\pm = (e_1 \pm ie_2)/\sqrt{2}, \qquad e_0 = e_3. \tag{A1.4}$$

The normalisation is such that

$$\frac{1}{4\pi}\int d\Omega_e\, Y^{l'}_{m'}(e)^* Y^l_m(e) = \delta_{ll'}\delta_{mm'}. \tag{A1.5}$$

The polynomials (A1.2) for given l may be written in terms of the $2l+1$ completely symmetric traceless lth-rank tensors t^l_m as

$$Y^l_m(e) = (1/l!)(t^l_m)_{j_1\ldots j_l} e_{j_1}\ldots e_{j_l}, \tag{A1.6}$$

and it follows from (A1.5) that

$$(1/l!)(t^l_m)^*_{j_1\ldots j_l}(t^l_{m'})_{j_1\ldots j_l} = [(2l+1)!/2^l l!]\delta_{mm'}. \tag{A1.7}$$

Since $Y^l_m(e)$ are homogeneous polynomials of degree l,

$$Y^l_m(ce) = c^l Y^l_m(e), \tag{A1.8}$$

(A1.2), (A1.3) or (A1.6) can be used to define spherical harmonics for a vector of arbitrary length.

The following expansion theorem will prove to be useful,

$$e^{k\cdot x} = \sum_{l=0}^{\infty} \phi_l(k^2x^2) \sum_{m=-l}^{l} Y^l_m(k)^* Y^l_m(x) = \sum_{l=0}^{\infty} \phi_l(k^2x^2)(|k|\,|x|)^l(2l+1)P_l\left(\frac{k\cdot x}{|k|\,|x|}\right), \tag{A1.9}$$

where ϕ_l is defined in terms of a spherical Bessel function j_l,

$$\phi_l(x) = \frac{j_l(-i\sqrt{x})}{(\sqrt{x})^l} = 2^l \sum_{n=0}^{\infty} \frac{(n+l)!}{(2n+2l+1)!n!} x^n. \tag{A1.10}$$

As a special instance of (A1.9) we have

$$e^{xy} = \sum_{l=0}^{\infty} (2l+1)x^l\phi_l(x^2)P_l(y). \tag{A1.11}$$

Appendix 2. Completeness relation for the generalised coherent states

We give the proof of the completeness relation (72). $k_l(|\xi|)$ is a modified spherical Bessel function

$$k_l(x)=(-1)^l x^l\left(\frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\right)^l\frac{\mathrm{e}^{-x}}{x}. \tag{A2.1}$$

We obtain

$$\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int\frac{\mathrm{d}^2\xi}{2\pi}\Phi_l(|\xi|)|\xi;l,m\rangle\langle\xi;l,m|$$

$$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=0}^{\infty}\frac{(n+l)!2^l}{(2n+2l+1)!n!}$$

$$\times\int_0^{\infty}\mathrm{d}|\xi|\,|\xi|^{2n+l+2}k_l(|\xi|)|l,m;l+2n\rangle\langle l,m;l+2n|$$

$$=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{n=0}^{\infty}|l,m;l+2n\rangle\langle l,m;l+2n|=1.$$

Appendix 3. Projection of generalised coherent states

Let $\boldsymbol{I}$ denote the angular momentum operator and define the Wigner D-functions by (Wigner 1959)

$$D^l_{m'm}(\boldsymbol{\theta})=\langle l,m'|\mathrm{e}^{-\mathrm{i}\boldsymbol{I}\cdot\boldsymbol{\theta}}|l,m\rangle. \tag{A3.1}$$

We then have the relations

$$D^l_{m0}(\alpha,\beta,\gamma)=\frac{1}{(2l+1)^{1/2}}Y^l_m(\beta,\alpha)^* \tag{A3.2}$$

and

$$Y^l_m(0,0)=\delta_{m0}(2l+1)^{1/2}, \tag{A3.3}$$

where (α,β,γ) are Euler angles of rotation. The D-functions obey the orthogonality relation

$$\int_0^{2\pi}\frac{\mathrm{d}\alpha}{2\pi}\int_0^{\pi}\frac{\mathrm{d}\beta\sin\beta}{2}\int_0^{2\pi}\frac{\mathrm{d}\gamma}{2\pi}D^{l_1}_{m'_1m_1}(\alpha,\beta,\gamma)^*D^{l_2}_{m'_2m_2}(\alpha,\beta,\gamma)$$

$$=\frac{1}{2l+1}\delta_{l_1l_2}\delta_{m'_1m'_2}\delta_{m_1m_2}. \tag{A3.4}$$

Since

$$Y^l_m(R(\alpha,\beta,0)f)=D^l_{mm'}(\alpha,\beta,0)^*Y^l_{m'}(f), \tag{A3.5}$$

we obtain, using (57), the result

$$\int_0^{2\pi}\frac{\mathrm{d}\alpha}{2\pi}\int_0^{\pi}\frac{\mathrm{d}\beta\sin\beta}{2}Y^l_m(\beta,\alpha)|R(\alpha,\beta,0)\boldsymbol{f}\rangle$$

$$=\mathrm{e}^{-\frac{1}{2}\boldsymbol{f}^*\cdot\boldsymbol{f}}\frac{Y^l_0(\boldsymbol{f})}{(2l+1)^{1/2}}\phi_l((\boldsymbol{f}\cdot\boldsymbol{f})\boldsymbol{a}^+\cdot\boldsymbol{a}^+)Y^l_m(\boldsymbol{a})^+|0\rangle, \tag{A3.6}$$

where we have used the relation

$$\int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} \int_0^{\pi} \frac{\mathrm{d}\beta \sin\beta}{2} Y^l_m(\beta, \alpha)^* D^{l'}_{m'm''}(\alpha, \beta, 0)$$
$$= \delta_{ll'}\delta_{mm'}\delta_{m''0}(2l+1)^{-1/2}. \tag{A3.7}$$

Now relations (50), (A3.3) and (67) can be used in (A3.6), and we finally obtain

$$\int_0^{2\pi} \frac{\mathrm{d}\alpha}{2\pi} \int_0^{\pi} \frac{\mathrm{d}\beta \sin\beta}{2} Y^l_m(\beta, \alpha)^* |R(\alpha, \beta, 0)f\rangle$$
$$= \mathrm{e}^{-\frac{1}{2}f^*\cdot f} \frac{Y^l_0(f)}{(2l+1)^{1/2}} (N_l(|f\cdot f|))^{-1} |f\cdot f;\, l,\, m\rangle, \tag{A3.8}$$

which is an eigenstate of A, $\boldsymbol{I}^2$ and $\boldsymbol{I}^3$.

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